MATH4240: Stochastic Processes Tutorial 8

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Recurrence of queuing chain

Let the number of customers arriving in unit time has density f and mean μ. Suppose that the chain is irreducible, then $f(0) > 0$ and $f(0) + f(1) < 1$, as shown in previous tutorial. We saw that the chain is recurrent if $\mu \leq 1$ and transient if $\mu > 1$.

Now, we will show

$$m_0 = \frac{1}{1-\mu},$$

where $m_0 = \mathbb{E}_0[T_0]$. It follows that if $\mu < 1$, then $m_0 < \infty$ and hence 0 is a positive recurrent state. By irreducibility, the chain is positive recurrent. On the other hand, if $\mu = 1$, then $m_0 = \infty$ and hence 0 is a null recurrent state. We conclude that an irreducible queuing chain is *positive recurrent* if $\mu < 1$, *null recurrent* if $\mu = 1$ and *transient* if $\mu > 1$.

Recurrence of queuing chain

We assume $f(0) > 0$, $f(0) + f(1) < 1$ and $\mu \leq 1$, so that the chain is irreducible and recurrent. Consider the chain start at a positive integer x. Then T_{x-1} denotes the time to go from state x to state $x - 1$, and $T_{y-1} - T_y$, $1 \leq y \leq x - 1$, denotes the time to go from state y to state $y - 1$. Since the queuing chain goes at most one step to the left at a time, the Markov property insures that the random variables

$$T_{x-1}, T_{x-2} - T_{x-1}, \dots, T_0 - T1$$

are independent. These random variables are identically distributed as

$$\min\{n > 0 : \xi_1 + \cdots + \xi_n = n - 1\},$$

i.e., as the smallest positive integer n such that the number of customers served by time n is one more than the number of new customers arriving by time n.

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Let $G(t), 0 \leq t \leq 1$ denote the probability generation function of the time to go from state 1 to state 0, i.e.

$$G(t) = \sum_{n=1}^{\infty} t^n P_1(T_0 = n).$$

The probability generating function of the sum of independent nonnegative integer-valued random variables is the product of their respective probability generating functions. If the chain starts at x, then

$$T_0 = T_{x-1} + (T_{x-2} - T_{x-1}) + \cdots + (T_0 - T_1)$$

is the sum of x independent random variables each having probability generating function $G(t)$.

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Thus, the probability generating function of T_0 is

$$(G(t))^x = \sum_{n=1}^{\infty} t^n P_x(T_0 = n).$$

We will now show that

$$G(t) = t\Phi(G(t)), \quad 0 \le t \le 1,$$

where Φ denotes the probability generating function of f.

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$$
\begin{aligned}
G(t) &= \sum_{n=1}^{\infty} t^n P_1(T_0 = n) \\
&= tP(1,0) + t\sum_{n=1}^{\infty} t^n P_1(T_0 = n+1) \\
&= tP(1,0) + t\sum_{n=1}^{\infty} t^n \sum_{y\neq 0} P(1,y) P_y(T_0 = n) \\
&= tP(1,0) + t\sum_{y\neq 0} P(1,y) \sum_{n=1}^{\infty} t^n P_y(T_0 = n) \\
&= tf(0) + t\sum_{y\neq 0} f(y)(G(t))^y \\
&= t\Phi(G(t)).
\end{aligned}
$$

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Now, we differentiate $G(t) = t\Phi(G(t))$ for $0 < t < 1$, and obtain

$$G'(t) = \Phi(G(t)) + tG'(t)\Phi'(G(t))$$

and

$$G'(t) = \frac{\Phi(G(t))}{1 - t\Phi'(G(t))}, 0 < t < 1.$$

Take $t \to 1$, we have $G(t) \to 1$ and $\Phi(t) \to 1$ and

$$\lim_{t\to 1} \Phi'(t) = \lim_{t\to 1} \sum_{x=1}^{\infty} xf(x)t^{x-1} = \sum_{x=1}^{\infty} xf(x) = \mu.$$

Thus,

$$\lim_{t\to 1} G'(t) = \lim_{t\to 1} \frac{\Phi(G(t))}{1 - t\Phi'(G(t))} = \frac{1}{1-\mu}.$$

Recurrence of queuing chain

Recall

$$G(t) := \sum_{n=1}^{\infty} P_1(T_0 = n)t^n.$$

Since $P(1,x) = P(0,x)$ for queuing chain. We have

$$G(t) = \sum_{n=1}^{\infty} P_0(T_0 = n)t^n$$

and hence

$$\begin{aligned} \frac{1}{1-\mu} = \lim_{t\to 1} G'(t) &= \lim_{t\to 1} \sum_{n=1}^{\infty} nP_0(T_0 = n)t^{n-1} \\ &= \sum_{n=1}^{\infty} nP_0(T_0 = n) \\ &= \mathbb{E}_0[T_0] = m_0. \end{aligned}$$

Decomposition of state space for periodic MC

Let X_n, $n \geq 0$, be an irreducible Markov chain with state space $\mathcal{S}$. The *period* is given by

$$d = g.c.d.\{n \geq 1 : P^n(x,x) > 0\}, \qquad x \in \mathcal{S},$$

where $g.c.d.$ means the *greatest common divisor* and d is independent of the choice of x.

Suppose that the chain is periodic ($d \geq 2$). Let $Y_m = X_{md}$, $m \geq 0$. Then Y_m, $m \geq 0$, is an aperiodic Markov chain with transition matrix $Q = P^d$. Moreover, the chain Y_m may be reducible and $\mathcal{S}$ is a union of d irreducible closed set:

$$\mathcal{S} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_d, \tag{1}$$

where for $i \in \{1, 2, \ldots, d\}$ and $x \in \mathcal{C}_i$ (we set $\mathcal{C}_{d+1} = \mathcal{C}_1$ here) we have

$$P(x,y) > 0 \quad \text{only if} \quad y \in \mathcal{C}_{i+1}.$$

Decomposition of state space for periodic MC

Example 0. Let $\mathcal{S} = \{1, 2\}$,

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ P = & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}.$$

Then,

$$P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

period $d = 2$ and $\mathcal{S} = \mathcal{C}_1 \cup \mathcal{C}_2$, with $\mathcal{C}_1 = \{1\}$ and $\mathcal{C}_2 = \{2\}$. Note that it has a stationary distribution $\pi = (1/2, 1/2)$.

Decomposition of state space for periodic MC

Example 1. Recall that in the last tutorial, we have a example:
$\mathcal{S} = \{1, 2, 3, 4\}$,

$$P = \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

Decomposition of state space for periodic MC

Note that

$$P^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 2/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 \end{pmatrix},$$

$$P^4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \end{pmatrix} = P.$$

Easily we get the period $d = 3$. Then $\mathcal{S} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ with

$$\mathcal{C}_1 = \{1, 4\}, \quad \mathcal{C}_2 = \{2\}, \quad \text{and} \quad \mathcal{C}_3 = \{3\}.$$

Example 2. $S = \{1, 2, 3, 4, 5, 6, 7\}$,

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Similarly we have $d = 3$ and $S = C_1 \cup C_2 \cup C_3$ with
$C_1 = \{1, 2\}, \quad C_2 = \{3, 4, 5\}, \quad C_3 = \{6, 7\}.$